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Pseudo Linear Pricing Rule for Utility Indifference Valuation

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Abstract

This paper considers exponential utility indifference pricing for a multidimensional non-traded assets model, and provides two linear approximations for the utility indifference price. The key tool is a probabilistic representation for the utility indifference price by the solution of a functional differential equation, which is termed *pseudo linear pricing rule*. We also provide an alternative derivation of the quadratic BSDE representation for the utility indifference price.

Keywords: utility indifference pricing, quadratic BSDE, FBSDE, functional differential equation.

Mathematics Subject Classification (2010): 91G40 91G80 60H30.

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1 Introduction

In this paper, we consider *exponential* utility indifference pricing in a multidimensional non-traded assets setting. Our interest is in pricing and hedging derivatives written on assets that are not traded. The market is incomplete as the risks arising from having exposure to non-traded assets cannot be fully hedged. There has been considerable research in the area of exponential utility indifference valuation, but despite the interest in this pricing and hedging approach, there have been relatively few explicit formulas available. The well known *one dimensional non-traded assets model* is an exception, and in a Markovian framework with a derivative written on a single non-traded asset, and partial hedging in a financial asset, Henderson and Hobson [23], Henderson [22], and Musiela and Zariphopoulou [33] use the Cole-Hopf transformation (or *distortion power*) to linearize the nonlinear partial differential equations (*PDEs* for short) for the value function. Subsequent generalizations of the model from Tehranchi [35], Frei and Schweizer [20] [21] have shown the exponential utility indifference price can still be written in a closed-form expression similar to that known for the Brownian setting, although the structure of the formula can be much less explicit. On the other hand, Davis [13] uses the duality to derive an explicit formula for the optimal hedging strategy, and Becherer [5] shows that the dual pricing formula exists even in a general semimartingale setting.

We study exponential utility indifference valuation in a multidimensional setting with the aim of developing a pricing methodology. The main tool that we use to prescribe the pricing dynamics is backward stochastic differential equation with quadratic growth (*quadratic BSDE* for short). It is well known in the literature that the utility indifference price can be written as a nonlinear expectation of the payoff under the original physical measure, and the nonlinear expectation is often specified by a quadratic BSDE. Several authors derive quadratic BSDE representations of exponential utility indifference values in models of varying generality - see Hu et al [25], Mania and Schweizer [30], Becherer [6], Morlais [32], Frei and Schweizer [20] [21], Bielecki and Jeanblanc [7], and Ankirchner et al [3] among others. Their derivations use the martingale optimality principle.

Our first contribution is to provide an alternative approach to derive the quadratic BSDE representation of the utility indifference price. The martingale optimality principle does not play any role in our derivation. Instead, we consider the associated utility maximization problems for utility indifference valuation from a risk-sensitive control perspective. We first transform our utility maximization problems into risk-sensitive control problems, and then use the comparison principle for a family of quadratic BSDEs indexed by the trading strategies to derive the pricing dynamics for the utility indifference price. The details are presented in Theorem 2.3. We call such a quadratic BSDE representation for the utility indifference price a *nonlinear pricing rule*.

With regards to the theory of quadratic BSDEs, the existence and uniqueness of bounded solutions was first proved in a Brownian setting by Kobylanski [28], and was extended to unbounded solutions and convex driver by Briand and Hu [9, 10], and Delbaen et al [15, 16]. The corresponding semimartingale case for bounded solutions may be found in Morlais [32] and Tevzadze [36], where in the former, the main theorems of [28] and [25] were extended, and in the latter, a fixed point argument with *BMO* martingale techniques was employed. See also Mocha and Westray [31] for the extension to unbounded solutions. In addition, Ankirchner et al [2] and Imkeller et al [26] consider the differentiability of quadratic BSDEs, and Frei et al [19] give convex bounds for the solutions. More recently, Barrieu and El Karoui [4] introduce a notion of quadratic semimartingale to study the stability of solutions, while Briand and Elie [8] find a simplified approach which was used to study the corresponding delayed equations. Finally, quadratic BSDEs with jumps were studied by

Becherer [6] for bounded solutions, and by El Karoui et al [17] for unbounded solutions.

Our main contribution is the provision of a new pricing formula for the utility indifference price, which we call a *pseudo linear pricing rule*. In Theorem 3.1, we represent the utility indifference price as a linear expectation of the payoff plus a pricing premium, where the latter is represented by the solution of a functional differential equation. Such an idea is motivated by Liang et al [29], where they transform BSDEs to functional differential equations, and solve them on a general filtered probability space. One of the advantages of such a representation is that we only need to solve a functional differential equation in order to calculate the utility indifference price. Moreover, the functional differential equation runs forwards, avoiding the conflicting nature between the backward equation and the underlying forward equation.

To apply such a *pseudo linear pricing rule*, we provide two linear approximations for the utility indifference price. In contrast to [29], where the driver is Lipschitz continuous, the driver of the functional differential equation considered in this paper is quadratic. Nevertheless, we can employ Picard iteration to approximate its solution. The first linear approximation is based on perturbations of the functional differential equation, the idea of which is motivated by Proposition 2 of Tevzadze [36]. The second linear approximation is based on a nonlinear version of Girsanov's transformation in order to vanish the driver. Such an idea has appeared in the BSDE literature, for example, as Proposition 11 in Mania and Schweizer [30] and measure solutions of BSDEs in Ankirchner et al [1], where they model the terminal data (the payoff) as a general random variable. In contrast, as we specify the dynamics of the underlying and the payoff structure, a coupled forward backward stochastic differential equation (*FBSDE* for short) appears naturally.

The paper is organized as follows: In Section 2, we present our multidimensional non-traded assets model, and present the nonlinear pricing rule, i.e. the quadratic BSDE representation for the utility indifference price. In Section 3, we present our pseudo linear pricing rule, i.e. the functional differential equation representation for the utility indifference price, and present two linear approximations for the utility indifference price based on such a representation formula.

2 Quadratic BSDEs and Nonlinear Pricing Rule

Let $\mathcal{W} = (W^1, \dots, W^d)$ be a d -dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ satisfying the *usual conditions*, where \mathcal{F}_t is the augmented σ -algebra generated by $(\mathcal{W}_u : 0 \leq u \leq t)$. The market consists of a traded financial index P , whose discounted price process is given by

$$P_t = P_0 + \int_0^t P_s(\mu_s^P ds + \langle \sigma_s^P, d\mathcal{W}_s \rangle), \quad (2.1)$$

and a set of observable but non-traded assets $\mathcal{S} = (S^1, \dots, S^d)$, whose discounted price processes are given by

$$S_t^i = S_0^i + \int_0^t S_s^i(\mu_s^i ds + \langle \sigma_s^i, d\mathcal{W}_s \rangle) \quad (2.2)$$

for $i = 1, \dots, d$. $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^d with its Euclidean norm $\|\cdot\|$. We have $\mu_t^P, \mu_t^i \in \mathbb{R}$, $\sigma_t^P = (\sigma_t^{P1}, \dots, \sigma_t^{Pd}) \in \mathbb{R}^d$ and $\sigma_t^i = (\sigma_t^{i1}, \dots, \sigma_t^{id}) \in \mathbb{R}^d$. There is also a risk-free bond or bank account with discounted price $B_t = 1$ for $t \geq 0$.

Our interest will be in pricing and hedging (path-dependent) contingent claims written on the non-traded assets \mathcal{S} . Specifically, we are concerned with contracts with the payoff at maturity T

of $g(\mathcal{S})$, which may depend on the whole path of \mathcal{S} . We impose the following assumptions, which will hold throughout:

- **Assumption (A1):** All the coefficients $\mu_t^i(\omega)$, $\sigma_t^i(\omega)$, $\mu_t^P(\omega)$ and $\sigma_t^P(\omega)$ are \mathcal{F}_t -predictable and uniformly bounded in (t, ω) .
- **Assumption (A2):** The volatility for the financial index P is uniformly elliptic: $\|\sigma_t^P(\omega)\| \geq \epsilon > 0$ for all (t, ω) .
- **Assumption (A3):** The payoff $g(\mathcal{S})$, as a functional of the stochastic process \mathcal{S} , is positive and bounded.

Our approach is to consider utility indifference valuation for such contingent claims. For a general overview of utility indifference valuation, we refer to the monograph edited by Carmona [11], and especially the survey article by Henderson and Hobson [24] therein. For this we need to consider the optimization problem for the investor both with and without the option. The investor has initial wealth $X_t \in \mathcal{F}_t$ at any starting time $t \in [0, T]$, and is able to trade the financial index with price P_t (and riskless bond with price 1). This will enable the investor to partially hedge the risks she is exposed to via her position in the claim.

The holder of the option has an exponential utility function with respect to her terminal wealth:

$$U_T(x) = -e^{-\gamma x} \quad \text{for } \gamma > 0 \text{ and } x \in \mathbb{R}.$$

The investor holds λ units of the claim, whose price at time $t \in [0, T]$ is denoted as \mathfrak{C}_t^λ and is to be determined, and invests her remaining wealth $X_t - \mathfrak{C}_t^\lambda$ in the financial index P . The investor will follow an admissible trading strategy

$$\begin{aligned} \pi \in \mathcal{A}_{ad}[0, T] &= \left\{ \pi : \pi \text{ is } \mathcal{F}_t\text{-predictable, } \sup_{\tau} \left\| E^{\mathbf{P}} \left[\int_{\tau}^T |\pi_t|^2 dt \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} < \infty \right. \\ &\quad \left. \text{for any } \mathcal{F}_t\text{-stopping time } \tau \in [0, T], \text{ and moreover,} \right. \\ &\quad \left. \epsilon \leq E^{\mathbf{P}} \left[e^{-\gamma \int_t^T \frac{\pi_s}{P_s} dP_s} \middle| \mathcal{F}_t \right] \leq K \text{ for a.e. } (t, \omega) \in [0, T] \times \Omega \right\}, \end{aligned}$$

for some constants $K \geq \epsilon > 0$, which results in the following wealth equation: For $0 \leq t \leq s \leq T$,

$$\begin{aligned} X_s^{X_t - \mathfrak{C}_t^\lambda}(\pi) &= X_t - \mathfrak{C}_t^\lambda + \int_t^s \frac{\pi_u}{P_u} dP_u \\ &= X_t - \mathfrak{C}_t^\lambda + \int_t^s \pi_u (\mu_u^P du + \langle \sigma_u^P, d\mathcal{W}_u \rangle). \end{aligned} \tag{2.3}$$

Remark 2.1 *The integrability conditions on the trading strategies π are slightly different from those required in Definition 1 of Hu et al [25]. They assume $E^{\mathbf{P}}[\int_0^T |\pi_t|^2 dt] < \infty$ to guarantee the no-arbitrage condition, and that the utility of the gain process $-e^{-\gamma \int_0^T \frac{\pi_s}{P_s} dP_s}$ is in Doob's class \mathcal{D} in order to apply the martingale optimality principle. Our first integrability condition is nothing but the BMO martingale property of $\int_0^\cdot \pi_s d\mathcal{W}_s$. The second condition is about the integrability of the utility of the remaining gain process $-e^{-\gamma \int_t^T \frac{\pi_s}{P_s} dP_s}$. Both of the integrability conditions are needed in order to derive the quadratic BSDE representation for the utility indifference price in*

the following Theorem 2.3. However, they are not restrictive if we only price and hedge contingent claims with bounded payoff, as the corresponding optimal trading strategy satisfies these conditions anyway, and coincides with the optimal trading strategy obtained in [25].

We recall a continuous martingale M with $E^{\mathbf{P}}[M, M]_T < \infty$ is called a **P-BMO** martingale if

$$\sup_{\tau} \|E^{\mathbf{P}}[|M_T - M_{\tau}|^2 | \mathcal{F}_{\tau}]\|_{\infty} < \infty$$

for any \mathcal{F}_t -stopping time $\tau \in [0, T]$. By Theorem 2.3 of Kazamaki [27], if M is a **P-BMO** martingale, its Doléans-Dade exponential $\mathcal{E}(M)$ is in Doob's class \mathcal{D} , and therefore uniformly integrable. Another useful property that will be used later is the following version of the John-Nirenburg inequality

$$\sup_{\tau} \|E^{\mathbf{P}}[|M_T - M_{\tau}|^2 | \mathcal{F}_{\tau}]\|_{\infty} \leq K_1 \sup_{\tau} \|E^{\mathbf{P}}[|M_T - M_{\tau}| | \mathcal{F}_{\tau}]\|_{\infty}^2 \quad (2.4)$$

for some constant $K_1 > 0$ (see Corollary 2.1 of [27]).

The investor will optimize over the admissible trading strategies to choose an optimal $\pi^{*,\lambda}$ by maximizing her expected terminal utility

$$\operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(X_T^{X_t - \mathfrak{C}_t^{\lambda}(\pi) + \lambda g(\mathcal{S}.)} \right)} \middle| \mathcal{F}_t \right]. \quad (2.5)$$

To define the utility indifference price for the option, we also need to consider the optimization problem for the investor without the option. This involves the investor investing only in the financial index itself. Her wealth equation is the same as (2.3) but starts from X_t , and she will choose an optimal $\pi^{*,0}$ by maximizing

$$\operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^{X_t}(\pi)} \middle| \mathcal{F}_t \right]. \quad (2.6)$$

We note that (2.6) is a special case of (2.5) with $\lambda = 0$.

Definition 2.2 (*Utility indifference valuation and hedging*)

The utility indifference price \mathfrak{C}_t^{λ} of λ units of the derivative with payoff $g(\mathcal{S}.)$ is defined by the solution to

$$\operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(X_T^{X_t - \mathfrak{C}_t^{\lambda}(\pi) + \lambda g(\mathcal{S}.)} \right)} \middle| \mathcal{F}_t \right] = \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^{X_t}(\pi)} \middle| \mathcal{F}_t \right].$$

The hedging strategy for λ units of the derivative is defined by the difference in the two optimal trading strategies $\pi^{*,\lambda} - \pi^{*,0}$.

The main result of this section is to show that the price of the option and the corresponding hedging strategy can be represented by the solution of a quadratic BSDE.

Theorem 2.3 (*Nonlinear pricing rule*)

Suppose that Assumptions (A1) (A2), and (A3) are satisfied. If $(Y^{\lambda}, \mathcal{Z}^{\lambda})$ is the unique solution of the following quadratic BSDE

$$Y_t^{\lambda} = \left(\lambda g(\mathcal{S}.) + \int_0^T \theta_s ds \right) + \int_t^T F_s(\mathcal{Z}_s^{\lambda}) ds - \int_t^T \langle \mathcal{Z}_s^{\lambda}, d\mathcal{W}_s \rangle, \quad (2.7)$$

with $\theta_s = \frac{|\mu_s^P|^2}{2\gamma\|\sigma_s^P\|^2}$, and the driver $F_s(z)$ given by

$$F_s(z) = -\frac{\gamma}{2}\|z\|^2 + \frac{\gamma}{2\|\sigma_s^P\|^2} \left| \langle \sigma_s^P, z \rangle - \frac{\mu_s^P}{\gamma} \right|^2 - \theta_s$$

for $z \in \mathbb{R}^d$, then the utility indifference price \mathfrak{C}_t^λ is represented by the solution of the quadratic BSDE (2.7)

$$\mathfrak{C}_t^\lambda = Y_t^\lambda - Y_t^0, \quad (2.8)$$

and the hedging strategy for λ units of the option is given by

$$-\frac{\langle \sigma_t^P, \mathcal{Z}_t^\lambda - \mathcal{Z}_t^0 \rangle}{\|\sigma_t^P\|^2}.$$

Remark 2.4 It is known that the above type of quadratic BSDE (2.7) can be derived by the martingale optimality principle - see, for example, Theorem 7 of Hu et al [25] and Section 3 of Ankirchner et al [3] in a Brownian motion setting, and Theorem 13 of Mania and Schweizer [30] and Section 2.1 of Morlais [32] in a general semimartingale setting. In the following, we provide a new proof of Theorem 2.3, where the martingale optimality principle does not play any role. Instead, we consider the problem from a risk-sensitive control perspective, and use the comparison principle for a family of quadratic BSDEs (2.11) indexed by the admissible trading strategy π to derive the quadratic BSDE representation. Although this technique is known in the literature (see Sections 19-21 of Quenez [34] and Section 3 of El Karoui et al [18]), we apply it for the first time in the context of quadratic BSDEs with unbounded random coefficients. On the other hand, treating utility indifference valuation from a risk-sensitive control viewpoint may also lead to new perspectives in utility maximization problems.

Proof. We consider the utility maximization problem (2.5). By using (2.3) in (2.5), we have

$$\begin{aligned} & \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[-e^{-\gamma(X_t - \mathfrak{C}_t^\lambda + \int_t^T \frac{\pi_s^s}{P_s^s} dP_s + \lambda g(S_\cdot))} \middle| \mathcal{F}_t \right] \\ &= -e^{-\gamma(X_t - \mathfrak{C}_t^\lambda)} \operatorname{ess\,inf}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[e^{-\gamma(\int_t^T \frac{\pi_s^s}{P_s^s} dP_s + \lambda g(S_\cdot))} \middle| \mathcal{F}_t \right] \\ &= -e^{-\gamma(X_t - \mathfrak{C}_t^\lambda)} \operatorname{ess\,inf}_{\pi \in \mathcal{A}_{ad}[t, T]} E^{\mathbf{P}} \left[e^{-\gamma(\int_t^T \frac{\pi_s^s}{P_s^s} dP_s - \theta_s ds)} e^{-\gamma(\lambda g(S_\cdot) + \int_0^T \theta_s ds)} \middle| \mathcal{F}_t \right] e^{\gamma \int_0^t \theta_s ds} \\ &= -e^{-\gamma(X_t - \mathfrak{C}_t^\lambda)} \exp \left\{ -\gamma \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} Y_t^\lambda(\pi) \right\} e^{\gamma \int_0^t \theta_s ds}, \end{aligned}$$

where $Y_t^\lambda(\pi)$ denotes the risk-sensitive control criterion

$$Y_t^\lambda(\pi) = -\frac{1}{\gamma} \ln E^{\mathbf{P}} \left[e^{-\gamma(\int_t^T \pi_s (\mu_s^P ds + \langle \sigma_s^P, dW_s \rangle) - \theta_s ds)} e^{-\gamma(\lambda g(S_\cdot) + \int_0^T \theta_s ds)} \middle| \mathcal{F}_t \right].$$

By Assumptions (A1)-(A3), $|\lambda g(\cdot) + \int_0^t \theta_s ds| \leq K_2$ for some constant $K_2 > 1$. Moreover, by the conditions on the admissible trading strategy π , we know that $Y_t^\lambda(\pi)$ is bounded for a.e. $(t, \omega) \in [0, T] \times \Omega$:

$$-\frac{1}{\gamma} \ln K - K_2 \leq Y_t(\pi) \leq -\frac{1}{\gamma} \ln \epsilon + K_2.$$

We further introduce the risk-sensitive control problem

$$\hat{Y}_t^\lambda = \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{ad}[t, T]} Y_t^\lambda(\pi).$$

In the following, we characterize both $Y^\lambda(\pi)$ and \hat{Y}^λ by the solutions of quadratic BSDEs.

First, we consider the risk-sensitive control criterion $Y^\lambda(\pi)$ under a different probability measure. For any given trading strategy $\pi \in \mathcal{A}_{ad}[0, T]$, we define a \mathbf{P} -BMO martingale

$$N_t(\pi) = - \int_0^t \gamma \pi_s \langle \sigma_s^P, d\mathcal{W}_s \rangle, \quad \text{for } t \in [0, T].$$

Indeed, for any \mathcal{F}_t -stopping time $\tau \in [0, T]$, by using the conditions on the admissible trading strategy π and Assumption (A1) we have

$$\begin{aligned} \sup_{\tau} \|E^{\mathbf{P}}[|N_T(\pi) - N_\tau(\pi)|^2 | \mathcal{F}_\tau]\|_\infty &= \sup_{\tau} \left\| E^{\mathbf{P}} \left[\int_\tau^T \gamma^2 |\sigma_s^P|^2 |\pi_s|^2 ds \middle| \mathcal{F}_\tau \right] \right\|_\infty \\ &\leq K \sup_{\tau} \left\| E^{\mathbf{P}} \left[\int_\tau^T |\pi_s|^2 ds \middle| \mathcal{F}_\tau \right] \right\|_\infty < \infty \end{aligned}$$

for some constant $K > 0$. Hence, the Doléans-Dade exponential $\mathcal{E}(N(\pi))$ is uniformly integrable, and we change the probability measure by defining

$$\frac{d\mathbf{P}^\pi}{d\mathbf{P}} = \mathcal{E}(N(\pi)) = \mathcal{E}\left(- \int_0^\cdot \gamma \pi_s \langle \sigma_s^P, d\mathcal{W}_s \rangle\right).$$

The risk-sensitive control criterion under \mathbf{P}^π becomes

$$\begin{aligned} Y_t^\lambda(\pi) &= -\frac{1}{\gamma} \ln E^{\mathbf{P}^\pi} \left[e^{-\int_t^T (\gamma \mu_s^P \pi_s - \frac{1}{2} \gamma^2 |\sigma_s^P|^2 |\pi_s|^2 - \gamma \theta_s) ds} e^{-\gamma(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds)} \middle| \mathcal{F}_t \right] \\ &= -\frac{1}{\gamma} \ln E^{\mathbf{P}^\pi} \left[e^{-\gamma \int_t^T F_s(\pi) ds} e^{-\gamma(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds)} \middle| \mathcal{F}_t \right], \end{aligned}$$

where we denote

$$F_s(\pi) = \mu_s^P \pi_s - \frac{\gamma}{2} |\sigma_s^P|^2 |\pi_s|^2 - \theta_s.$$

Note that $F_s(\pi)$ only depends on π_s , not on all of π .

Next, we characterize $Y^\lambda(\pi)$ by the solution of a quadratic BSDE with unbounded random coefficients, whose existence is proved by directly showing that $Y^\lambda(\pi)$ indeed satisfies this BSDE. Indeed, note that for $t \in [0, T]$,

$$\bar{Y}_t^\lambda(\pi) = e^{-\gamma(Y_t^\lambda(\pi) + \int_0^t F_s(\pi) ds)}$$

is a uniformly integrable martingale under \mathbf{P}^π , since

$$e^{-\gamma(Y_t^\lambda(\pi) + \int_0^t F_s(\pi) ds)} = E^{\mathbf{P}^\pi} \left[e^{-\gamma \int_0^T F_s(\pi) ds} e^{-\gamma(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds)} \middle| \mathcal{F}_t \right].$$

By the martingale representation theorem, there exists an \mathcal{F}_t -predictable process $\bar{\mathcal{Z}}^\lambda(\pi)$ such that

$$\bar{Y}_t^\lambda(\pi) = e^{-\gamma(\lambda g(\mathcal{S}_.) + \int_0^T (\theta_s + F_s(\pi)) ds)} - \int_t^T \langle \bar{\mathcal{Z}}_s^\lambda(\pi), d\mathcal{W}_s(\pi) \rangle, \quad (2.9)$$

where $\mathcal{W}(\pi) = \mathcal{W} - [\mathcal{W}, N(\pi)]$ is Brownian motion under \mathbf{P}^π by Girsanov's transformation.

For any $t \in [0, T]$, if we define $\mathcal{Z}_t^\lambda(\pi) = -\frac{1}{\gamma} \bar{\mathcal{Z}}_t^\lambda(\pi) / \bar{Y}_t^\lambda(\pi)$, and apply Itô's formula to $Y_t^\lambda(\pi) = -\frac{1}{\gamma} \ln \bar{Y}_t^\lambda(\pi) - \int_0^t F_s(\pi) ds$, then it is easy to verify that $(Y^\lambda(\pi), \mathcal{Z}^\lambda(\pi))$ is a solution to the following quadratic BSDE

$$\begin{aligned} Y_t^\lambda(\pi) = & \left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) \\ & + \int_t^T \left(F_s(\pi) - \frac{\gamma}{2} \|\mathcal{Z}_s^\lambda(\pi)\|^2 \right) ds - \int_t^T \langle \mathcal{Z}_s^\lambda(\pi), d\mathcal{W}_s(\pi) \rangle. \end{aligned} \quad (2.10)$$

Equivalently under the original probability measure \mathbf{P} , we write

$$\begin{aligned} Y_t^\lambda(\pi) = & \left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) \\ & + \int_t^T \left(F_s(\pi) - \gamma \langle \sigma_s^P, \mathcal{Z}_s^\lambda(\pi) \rangle \pi_s - \frac{\gamma}{2} \|\mathcal{Z}_s^\lambda(\pi)\|^2 \right) ds - \int_t^T \langle \mathcal{Z}_s^\lambda(\pi), d\mathcal{W}_s \rangle. \end{aligned} \quad (2.11)$$

We notice that BSDE (2.11) has quadratic growth in z with unbounded random coefficients, which satisfy the *BMO* condition in Theorem 8 of Mania and Schweizer [30]. Since the solution $Y^\lambda(\pi)$ is bounded, Theorem 8 of [30] then implies that a comparison theorem holds for (2.11). Let $\pi^1, \pi^2 \in \mathcal{A}_{ad}[0, T]$ such that

$$F_s(\pi^1) - \gamma \langle \sigma_s^P, z \rangle \pi_s^1 \geq F_s(\pi^2) - \gamma \langle \sigma_s^P, z \rangle \pi_s^2$$

for $z \in \mathbb{R}^d$. Then $Y_t^\lambda(\pi^1) \geq Y_t^\lambda(\pi^2)$ for *a.e.* (t, ω) . This can be proved either by changing probability measure as in [30], or by an exponential change of variables.

As a byproduct, we also obtain that the quadratic BSDE (2.11) admits a unique solution $(Y^\lambda(\pi), \mathcal{Z}^\lambda(\pi))$, where $Y^\lambda(\pi)$ is a bounded special semimartingale with $\mathcal{Z}^\lambda(\pi)$ as its corresponding martingale representation.

Thirdly, we prove that the solution of our risk-sensitive control problem is given by

$$\hat{Y}_t^\lambda = Y_t^\lambda, \quad (2.12)$$

and the optimal trading strategy is given by

$$\pi_s^{*, \lambda} = -\frac{\langle \sigma_s^P, \mathcal{Z}_s^\lambda \rangle}{\|\sigma_s^P\|^2} + \frac{\mu_s^P}{\gamma \|\sigma_s^P\|^2} \quad (2.13)$$

for $s \in [t, T]$, where $(Y^\lambda, \mathcal{Z}^\lambda)$ solves the quadratic BSDE (2.5), whose existence and uniqueness is guaranteed by Theorems 2.3 and 2.6 of Kobylanski [28] or Theorem 1 of Tevzadze [36]. Indeed, the

driver of (2.5) satisfies $F_t(0) = 0$, and is smooth in z with

$$\begin{aligned}\nabla_z F_t(z) &= -\gamma z + \frac{\gamma}{\|\sigma_t^P\|^2} \left(\langle \sigma_t^P, z \rangle - \frac{\mu_t^P}{\gamma} \right) \sigma_t^P, \\ \nabla_{zz} F_t(z) &= -\gamma \mathbf{1} + \frac{\gamma}{\|\sigma_t^P\|^2} (\sigma_t^P)^T \sigma_t^P,\end{aligned}$$

where the superscript T denotes the matrix transposition. Hence, by Assumptions (A1)-(A3),

$$\begin{aligned}\|\nabla_z F_t(z)\| &\leq K_2(1 + \|z\|), \\ \|z\|^2/K_2 &\leq z \nabla_{zz} F_t(z) z^T \leq K_2 \|z\|^2,\end{aligned}\tag{2.14}$$

and the terminal data satisfies

$$\left| \lambda g(\cdot) + \int_0^T \theta_s ds \right| \leq K_2 \tag{2.15}$$

for some constant $K_2 \geq 1$. Therefore, there exists a unique solution $(Y^\lambda, \mathcal{Z}^\lambda)$ to BSDE (2.5), where Y^λ is a bounded special semimartingale with \mathcal{Z}^λ as its martingale representation. Moreover, the martingale part $\int_0^\cdot \langle \mathcal{Z}^\lambda, d\mathcal{W} \rangle$ is a **P**-BMO martingale.

Now we proceed to prove (2.12) and (2.13). Notice that for any $\pi \in \mathcal{A}_{ad}[t, T]$,

$$F_s(\pi) - \gamma \langle \sigma_s^P, \mathcal{Z}_s^\lambda \rangle \pi_s - \frac{\gamma}{2} \|\mathcal{Z}_s^\lambda\|^2 = -\frac{\gamma}{2} \|\sigma_s^P\|^2 |\pi_s - \pi_s^{*,\lambda}|^2 + F_s(\mathcal{Z}_s^\lambda) \leq F_s(\mathcal{Z}_s^\lambda),$$

and for $\pi = \pi^{*,\lambda}$,

$$F_s(\pi^{*,\lambda}) - \gamma \langle \sigma_s^P, \mathcal{Z}_s^\lambda \rangle \pi_s^{*,\lambda} - \frac{\gamma}{2} \|\mathcal{Z}_s^\lambda\|^2 = F_s(\mathcal{Z}_s^\lambda).$$

If $\pi^{*,\lambda} \in \mathcal{A}_{ad}[t, T]$, then by applying the comparison theorem for the quadratic BSDE (2.7), we obtain that $Y_t^\lambda(\pi) \leq Y_t^\lambda$ for any $\pi \in \mathcal{A}_{ad}[t, T]$, and $Y_t^\lambda(\pi^{*,\lambda}) = Y_t^\lambda$.

Since $\hat{Y}_t^\lambda = \sup_{\pi \in \mathcal{A}_{ad}[t, T]} Y_t^\lambda(\pi)$, we get $\hat{Y}_t^\lambda = Y_t^\lambda$ and $\pi^{*,\lambda}$ achieves the maximum. We are left to verify that $\pi^{*,\lambda} \in \mathcal{A}_{ad}[t, T]$. For this, we only need to note that $\int_0^\cdot \langle \mathcal{Z}_s^\lambda, d\mathcal{W}_s \rangle$ is a **P**-BMO martingale, and

$$Y_t^\lambda = Y_t^\lambda(\pi^{*,\lambda}) = -\frac{1}{\gamma} \ln E^{\mathbf{P}} \left[e^{-\gamma \int_t^T \frac{\pi_s^{*,\lambda}}{\|\sigma_s^P\|^2} dP_s} e^{-\gamma \lambda g(S_\cdot)} \middle| \mathcal{F}_t \right]$$

is bounded for *a.e.* $(t, \omega) \in [0, T] \times \Omega$, so is $E^{\mathbf{P}}[e^{-\gamma \int_t^T \frac{\pi_s^{*,\lambda}}{\|\sigma_s^P\|^2} dP_s} | \mathcal{F}_t]$.

The optimization problem (2.6) is a special case of (2.5) with $\lambda = 0$. Finally, by Definition 2.2, the price \mathfrak{C}_t^λ is given by the solution to

$$\begin{aligned}-e^{-\gamma(X_t - \mathfrak{C}_t^\lambda + Y_t^\lambda)} e^{\gamma \int_0^t \theta_s ds} &= E^{\mathbf{P}} \left[-e^{-\gamma(X_T^{X_t - \mathfrak{C}_t^\lambda}(\pi^{*,\lambda}) + \lambda g(S_\cdot))} \middle| \mathcal{F}_t \right] \\ &= E^{\mathbf{P}} \left[-e^{-\gamma X_T^{X_t}(\pi^{*,0})} \middle| \mathcal{F}_t \right] = -e^{-\gamma(X_t + Y_t^0)} e^{\gamma \int_0^t \theta_s ds}.\end{aligned}$$

Therefore, $\mathfrak{C}_t^\lambda = Y_t^\lambda - Y_t^0$, and the hedging strategy for λ units of the option is given by

$$\pi_t^{*,\lambda} - \pi_t^{*,0} = -\frac{\langle \sigma_t^P, \mathcal{Z}_t^\lambda \rangle}{\|\sigma_t^P\|^2} + \frac{\mu_t^P}{\gamma \|\sigma_t^P\|^2} + \frac{\langle \sigma_t^P, \mathcal{Z}_t^0 \rangle}{\|\sigma_t^P\|^2} - \frac{\mu_t^P}{\gamma \|\sigma_t^P\|^2} = -\frac{\langle \sigma_t^P, \mathcal{Z}_t^\lambda - \mathcal{Z}_t^0 \rangle}{\|\sigma_t^P\|^2},$$

which completes the proof. \blacksquare

3 Functional Differential Equations and Pseudo Linear Pricing Rule

In this section, we present our pseudo linear pricing rule for the utility indifference price \mathfrak{C}_t^λ . The main idea is motivated by Liang et al [29], where the authors introduce a class of functional differential equations in order to solve BSDEs on a general filtered probability space. See also Casserini and Liang [12] for a generalization of this method to solve FBSDEs. The solution Y to BSDE (2.7) is a bounded special semimartingale, so admits a unique decomposition under \mathbf{P} :

$$Y_t^\lambda = M_t^{\lambda, \mathbf{P}} - V_t^{\lambda, \mathbf{P}}, \quad \text{for } t \in [0, T],$$

where $M^{\lambda, \mathbf{P}}$ is the martingale part, which is a \mathbf{P} -BMO martingale, and $V^{\lambda, \mathbf{P}}$ is the finite variation part with $V_0^{\lambda, \mathbf{P}} = 0$. By the martingale property of $M^{\lambda, \mathbf{P}}$,

$$\begin{aligned} Y_t^\lambda &= E^\mathbf{P} \left[Y_T^\lambda + V_T^{\lambda, \mathbf{P}} \middle| \mathcal{F}_t \right] - V_t^{\lambda, \mathbf{P}} \\ &= E^\mathbf{P} \left[\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \middle| \mathcal{F}_t \right] + E^\mathbf{P} [V_T^{\lambda, \mathbf{P}} - V_t^{\lambda, \mathbf{P}} | \mathcal{F}_t]. \end{aligned} \quad (3.1)$$

In other words, knowing the finite variation process $V^{\lambda, \mathbf{P}}$ and the terminal data Y_T^λ is enough to calculate Y^λ , which in turn gives us a new pricing rule for the utility indifference price \mathfrak{C}_t^λ .

Theorem 3.1 (*Pseudo linear pricing rule*)

Suppose that Assumptions (A1) (A2), and (A3) are satisfied. If $V^{\lambda, \mathbf{P}}$ is the unique solution of the following functional differential equation

$$V_t^{\lambda, \mathbf{P}} = \int_0^t F_s(\mathcal{Z}_s^{\lambda, \mathbf{P}}(V^{\lambda, \mathbf{P}})) ds, \quad (3.2)$$

with $\mathcal{Z}^{\lambda, \mathbf{P}}(\cdot)$, as an affine functional of a stochastic process V , given by

$$\begin{aligned} \int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{P}}(V), dW_s \rangle &= \left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) + V_T \\ &\quad - E^\mathbf{P} \left[\left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) + V_T \middle| \mathcal{F}_t \right], \end{aligned}$$

then the utility indifference price \mathfrak{C}_t^λ can be represented by the following linear conditional expectation

$$\mathfrak{C}_t^\lambda = E^\mathbf{P} [\lambda g(\mathcal{S}_.) | \mathcal{F}_t] + E^\mathbf{P} [V_T^{\lambda, \mathbf{P}} - V_t^{\lambda, \mathbf{P}} | \mathcal{F}_t] - E^\mathbf{P} [V_T^{0, \mathbf{P}} - V_t^{0, \mathbf{P}} | \mathcal{F}_t], \quad (3.3)$$

and the hedging strategy for λ units of the option is given by

$$-\frac{\langle \sigma_t^P, \mathcal{Z}_t^{\lambda, \mathbf{P}}(V^{\lambda, \mathbf{P}}) - \mathcal{Z}_t^{0, \mathbf{P}}(V^{0, \mathbf{P}}) \rangle}{\|\sigma_t^P\|^2}.$$

Proof. To obtain the functional differential equation (3.2), we take conditional expectation of (2.7) on \mathcal{F}_t :

$$\begin{aligned} Y_t^\lambda &= E^{\mathbf{P}} \left[\left(\lambda g(\mathcal{S}) + \int_0^T \theta_s ds \right) + \int_t^T F_s(Z_s^\lambda) ds \middle| \mathcal{F}_t \right] \\ &= E^{\mathbf{P}} \left[\left(\lambda g(\mathcal{S}) + \int_0^T \theta_s ds \right) + \int_0^T F_s(Z_s^\lambda) ds \middle| \mathcal{F}_t \right] - \int_0^t F_s(Z_s^\lambda) ds. \end{aligned}$$

On the other hand, Y^λ admits the decomposition $Y_t^\lambda = M_t^{\lambda, \mathbf{P}} - V_t^{\lambda, \mathbf{P}}$. Due to the uniqueness of special semimartingale decomposition, we obtain (3.2) by identifying the finite variation parts of the above two expressions for Y^λ . To show that $Z_t^\lambda = Z_t^{\lambda, \mathbf{P}}(V^{\lambda, \mathbf{P}})$, we only need to note that Z^λ is the martingale representation of $M^{\lambda, \mathbf{P}}$. Finally, (3.3) follows from (2.8) and (3.1). ■

Since (3.3) is under linear expectation, we call it the *pseudo linear pricing rule* for the utility indifference price \mathfrak{C}_t^λ . The advantage of this pricing rule compared to the nonlinear pricing rule (2.8) is that we only need to solve functional differential equation (3.2) for $V^{\lambda, \mathbf{P}}$ in order to calculate the utility indifference price \mathfrak{C}_t^λ . Moreover, functional differential equation (3.2) runs forwards. Thus, the conflicting nature between the backward equation and the underlying forward equation is avoided.

Whereas in [29] the driver is Lipschitz continuous, the driver $F_t(z)$ of functional differential equation (3.2) is quadratic in z . Nevertheless, we can employ Picard iteration to approximate $V^{\lambda, \mathbf{P}}$, and therefore the utility indifference price \mathfrak{C}_t^λ . We rely mainly on the change of probability measure, and we will present two linear approximations for \mathfrak{C}_t^λ depending on different choices of probability measures. We first present an equivalent formulation of pseudo linear pricing rule but under a different probability measure.

Corollary 3.2 *Let $\mathcal{B} = (B^1, \dots, B^d)$ be a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$. Suppose that N is some \mathbf{Q} -BMO martingale, so that its Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable. Define*

$$\frac{d\mathbf{P}}{d\mathbf{Q}} = \mathcal{E}(N).$$

Suppose that $V^{\lambda, \mathbf{Q}}$ solves the following functional differential equation

$$V_t^{\lambda, \mathbf{Q}} = \int_0^t F_s(Z_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}})) ds + \langle Z_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d[\mathcal{B}, N]_s \rangle, \quad (3.4)$$

with $Z^{\lambda, \mathbf{Q}}(\cdot)$, as an affine functional of V , given by

$$\begin{aligned} \int_t^T \langle Z_s^{\lambda, \mathbf{Q}}(V), d\mathcal{B}_s \rangle &= \left(\lambda g(\mathcal{S}) + \int_0^T \theta_s ds \right) + V_T \\ &- E^{\mathbf{Q}} \left[\left(\lambda g(\mathcal{S}) + \int_0^T \theta_s ds \right) + V_T \middle| \mathcal{F}_t \right], \end{aligned}$$

and $\mathcal{S} = (S^1, \dots, S^d)$ given by

$$S_t^i = S_0^i + \int_0^t S_s^i (\mu_s^i ds - \langle \sigma_s^i, d[\mathcal{B}, N]_s \rangle + \langle \sigma_s^i, d\mathcal{B}_s \rangle). \quad (3.5)$$

Then

$$V_t^{\lambda, \mathbf{P}} = V_t^{\lambda, \mathbf{Q}} - \int_0^t \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d[\mathcal{B}, N]_s \rangle \quad (3.6)$$

solves (3.2) on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$.

Proof. By the definition of $V^{\lambda, \mathbf{P}}$ in (3.6) and functional differential equation (3.4), we have

$$V_t^{\lambda, \mathbf{P}} = \int_0^t F_s(\mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}})) ds.$$

Hence, we only need to show that $\mathcal{Z}_t^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}) = \mathcal{Z}_t^{\lambda, \mathbf{P}}(V^{\lambda, \mathbf{P}})$ for $t \in [0, T]$, which means $\mathcal{Z}^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}})$ is invariant under the change of probability measure. In other words, the martingale representation is invariant under the change of probability measure:

$$\int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d\mathcal{W}_s \rangle = M_T^{\lambda, \mathbf{P}} - M_t^{\lambda, \mathbf{P}},$$

with

$$M_t^{\lambda, \mathbf{P}} = E^{\mathbf{P}} \left[\left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) + V_T^{\lambda, \mathbf{P}} \middle| \mathcal{F}_t \right],$$

and $\mathcal{W} = \mathcal{B} - [\mathcal{B}, N]$ being Brownian motion under the probability measure \mathbf{P} . Indeed, using Girsanov's transformation,

$$\begin{aligned} & \int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d\mathcal{W}_s \rangle \\ &= \int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d\mathcal{B}_s \rangle - \int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d[\mathcal{B}, N]_s \rangle \\ &= \left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) + V_T^{\lambda, \mathbf{Q}} - E^{\mathbf{Q}} \left[\left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) + V_T^{\lambda, \mathbf{Q}} \middle| \mathcal{F}_t \right] \\ &\quad - \int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d[\mathcal{B}, N]_s \rangle \\ &= \left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) + V_T^{\lambda, \mathbf{P}} - E^{\mathbf{P}} \left[\left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) + V_T^{\lambda, \mathbf{P}} \middle| \mathcal{F}_t \right] \\ &= M_T^{\lambda, \mathbf{P}} - M_t^{\lambda, \mathbf{P}}. \end{aligned}$$

■

3.1 Perturbations of Functional Differential Equations

By the pseudo linear pricing rule (3.3), we only need to solve functional differential equation (3.2) for $V^{\lambda, \mathbf{P}}$ in order to obtain the utility indifference price \mathfrak{C}_t^λ . Our first linear approximation for the utility indifference price \mathfrak{C}_t^λ is based on perturbations of functional differential equation (3.2), the idea of which is motivated by Proposition 2 of Tevzadze [36]¹.

¹We thank one of the referees for the suggestion of this method.

We first decompose the units of the option λ as the following finite sum: $\lambda = \sum_{j=1}^J \lambda_j$ such that

$$\lambda_j \leq \frac{\lambda}{32K_1K_2^2},$$

where K_1 is the constant from the John-Nirenburg inequality (2.4), and K_2 is the constant from BSDE (2.7) (see (2.14) and (2.15)). We then make perturbations of functional differential equation (3.2) as follows:

$$V_t^{\lambda_j, \mathbf{P}} = \int_0^t F_s \left(\sum_{k=1}^j \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right) - F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right) ds, \quad (3.7)$$

with $\mathcal{Z}^{\lambda_j, \mathbf{P}}(\cdot)$, as an affine functional of V , given by

$$\begin{aligned} \int_t^T \langle \mathcal{Z}_s^{\lambda_j, \mathbf{P}}(V), d\mathcal{W}_s \rangle &= \frac{\lambda_j}{\lambda} \left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) + V_T \\ &\quad - E^{\mathbf{P}} \left[\frac{\lambda_j}{\lambda} \left(\lambda g(\mathcal{S}_.) + \int_0^T \theta_s ds \right) + V_T \middle| \mathcal{F}_t \right], \end{aligned}$$

and with $\sum_{k=1}^0 = 0$ by convention. Then it is easy to verify that $V_t^{\lambda, \mathbf{P}} = \sum_{j=1}^J V_t^{\lambda_j, \mathbf{P}}$ solves functional differential equation (3.2). For functional differential equation (3.7), we can give the following linear approximation for its solution $V^{\lambda_j, \mathbf{P}}$.

Define the Banach space $\mathfrak{V}([0, T]; \mathbb{R})$ for the continuous and \mathcal{F}_t -adapted processes valued in \mathbb{R} , endowed with the norm

$$\|V\|_{\mathfrak{V}[0, T]} = \sup_{\tau} \|E[V_T - V_{\tau} | \mathcal{F}_{\tau}]\|_{\infty}$$

for any \mathcal{F}_t -stopping time $\tau \in [0, T]$. Furthermore, define its subspace

$$\mathfrak{V}([0, T]; B_r) = \left\{ V \in \mathfrak{V}([0, T]; \mathbb{R}) : \|V\|_{\mathfrak{V}[0, T]} \leq r \text{ for } r = \frac{1}{32K_1K_2} \right\}.$$

Proposition 3.3 *Let $\mathcal{B} = (B^1, \dots, B^d)$ be a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$. For fixed j where $1 \leq j \leq J$, suppose that we have solved functional differential equation (3.7) and obtained its solution $V^{\lambda_k, \mathbf{P}}$ for $k = 1, \dots, j-1$, so that we have the affine functionals $\mathcal{Z}^{\lambda_k, \mathbf{P}}(\cdot)$. Then N^j defined by*

$$N^j = \int_0^{\cdot} \langle \nabla_z F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right), d\mathcal{B}_s \rangle$$

is a \mathbf{Q} -BMO martingale.

Define the following sequence $\{V^{\lambda_j, \mathbf{Q}}(m)\}_{m \geq 0}$ iteratively: $V^{\lambda_j, \mathbf{Q}}(0) = 0$,

$$V_t^{\lambda_j, \mathbf{Q}}(m+1) = \int_0^t \tilde{F}_s(\mathcal{Z}_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m))) ds,$$

with $\tilde{F}_s(z)$ given by

$$\begin{aligned}\tilde{F}_s^j(z) &= F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) + z \right) - F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right) \\ &\quad - \langle \nabla_z F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right), z \rangle.\end{aligned}$$

Then $\{V^{\lambda_j, \mathbf{Q}}(m)\}_{m \geq 0}$ converges to some $V^{\lambda_j, \mathbf{Q}}$ in the space $\mathfrak{V}([0, T]; B_r)$ with the convergence rate

$$\|V^{\lambda_j, \mathbf{Q}} - V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{V}[0, T]} \leq r \left(\frac{1}{2} \right)^{m-1},$$

and $V^{\lambda_j, \mathbf{P}}$ is obtained by (3.6):

$$V_t^{\lambda_j, \mathbf{P}} = V_t^{\lambda_j, \mathbf{Q}} - \int_0^t \langle \mathcal{Z}_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}), \nabla_z F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) \right) \rangle ds.$$

Proof. For fixed j , we first verify that N^j is a \mathbf{Q} -BMO martingale. By Corollary 3.2, we have

$$\mathcal{Z}_s^{\lambda_k, \mathbf{P}}(V^{\lambda_k, \mathbf{P}}) = \mathcal{Z}_s^{\lambda_k, \mathbf{Q}}(V^{\lambda_k, \mathbf{Q}})$$

for $k = 1, \dots, j-1$. Therefore,

$$\begin{aligned}& \sup_{\tau} \left\| E^{\mathbf{Q}}[|N_T^j - N_{\tau}^j|^2 | \mathcal{F}_{\tau}] \right\|_{\infty} \\ &= \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\int_{\tau}^T \left| \nabla_z F_s \left(\sum_{k=1}^{j-1} \mathcal{Z}_s^{\lambda_k, \mathbf{Q}}(V^{\lambda_k, \mathbf{Q}}) \right) \right|^2 ds \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} \\ &\leq \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\int_{\tau}^T 2K_2^2 (1 + \|\mathcal{Z}_s^{\lambda_k, \mathbf{Q}}(V^{\lambda_k, \mathbf{Q}})\|^2) ds \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} \\ &< \infty,\end{aligned}$$

where we used (2.14) and the \mathbf{Q} -BMO martingale property of

$$\int_0^{\cdot} \langle \mathcal{Z}_s^{\lambda_k, \mathbf{Q}}(V^{\lambda_k, \mathbf{Q}}), d\mathcal{B}_s \rangle.$$

Next, we consider the convergence of the sequence $\{V^{\lambda_j, \mathbf{Q}}(m)\}_{m \geq 0}$. Similar to Remark 1 of [36], by using the mean value theorem twice on $\tilde{F}_s^j(\cdot)$ and by using (2.14), it is easy to verify that

$$\begin{aligned}|\tilde{F}_s^j(z) - \tilde{F}_s^j(\bar{z})| &\leq K_2(\|z\| + \|\bar{z}\|)\|z - \bar{z}\|, \\ |\tilde{F}_s^j(z)| &\leq K_2\|z\|^2.\end{aligned}$$

Given that $V^{\lambda_j, \mathbf{Q}}(m) \in \mathfrak{V}([0, T]; B_r)$, we need to verify that $V^{\lambda_j, \mathbf{Q}}(m+1)$ is in the same space $\mathfrak{V}([0, T]; B_r)$. Indeed,

$$\begin{aligned}
& \|V^{\lambda_j, \mathbf{Q}}(m+1)\|_{\mathfrak{V}[0, T]} \\
&= \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_{\tau}^T \tilde{F}_s(Z_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m))) ds \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} \\
&\leq K_2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\int_{\tau}^T \|Z_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m))\|^2 ds \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} \\
&= K_2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_{\tau}^T \langle Z_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m)), d\mathcal{B}_s \rangle \right|^2 \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty} \\
&\leq K_1 K_2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_{\tau}^T \langle Z_s^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m)), d\mathcal{B}_s \rangle \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2, \tag{3.8}
\end{aligned}$$

where we used the John-Nirenburg inequality (2.4) in the last inequality. In the following, we denote

$$\xi_j = \frac{\lambda_j}{\lambda} \left(\lambda g(\mathcal{S}) + \int_0^T \theta_s ds \right).$$

By the definition of λ_j and (2.15),

$$|\xi_j| \leq \frac{1}{32K_1K_2}.$$

With the notation ξ_j , the affine functional $Z^{\lambda_j, \mathbf{Q}}(\cdot)$ is rewritten as

$$\int_t^T \langle Z_s^{\lambda_j, \mathbf{Q}}(V), d\mathcal{B}_s \rangle = \xi_j + V_T - E^{\mathbf{Q}}[\xi_j + V_T | \mathcal{F}_t].$$

Therefore, following (3.8), $\|V^{\lambda_j, \mathbf{Q}}(m+1)\|_{\mathfrak{V}[0, T]}$ is further dominated by

$$\begin{aligned}
& K_1 K_2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \xi_j + V_T^{\lambda_j, \mathbf{Q}}(m) - E^{\mathbf{Q}}[\xi_j + V_T^{\lambda_j, \mathbf{Q}}(m) | \mathcal{F}_{\tau}] \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2 \\
&= K_1 K_2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \xi_j - E^{\mathbf{Q}}[\xi_j | \mathcal{F}_{\tau}] + V_T^{\lambda_j, \mathbf{Q}}(m) - V_{\tau}^{\lambda_j, \mathbf{Q}}(m) \right. \right. \right. \\
&\quad \left. \left. - E^{\mathbf{Q}}[V_T^{\lambda_j, \mathbf{Q}}(m) - V_{\tau}^{\lambda_j, \mathbf{Q}}(m) | \mathcal{F}_{\tau}] \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2 \\
&\leq 4K_1 K_2 \left(\sup_{\tau} \|E^{\mathbf{Q}}[\xi_j | \mathcal{F}_{\tau}]\|_{\infty}^2 + \sup_{\tau} \|E^{\mathbf{Q}}[|E^{\mathbf{Q}}(\xi_j | \mathcal{F}_{\tau})| | \mathcal{F}_{\tau}]\|_{\infty}^2 \right. \\
&\quad \left. + \sup_{\tau} \|E^{\mathbf{Q}}[|V_T^{\lambda_j, \mathbf{Q}}(m) - V_{\tau}^{\lambda_j, \mathbf{Q}}(m)| | \mathcal{F}_{\tau}]\|_{\infty}^2 \right. \\
&\quad \left. + \sup_{\tau} \|E^{\mathbf{Q}}[|E^{\mathbf{Q}}(V_T^{\lambda_j, \mathbf{Q}}(m) - V_{\tau}^{\lambda_j, \mathbf{Q}}(m) | \mathcal{F}_{\tau})| | \mathcal{F}_{\tau}]\|_{\infty}^2 \right) \\
&\leq 8K_1 K_2 \left(|\xi_j|^2 + \|V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{V}[0, T]}^2 \right) \leq 1/(64K_1 K_2) \leq r.
\end{aligned}$$

Similarly, we consider the difference $\delta V^{\lambda_j, \mathbf{Q}}(m) = V^{\lambda_j, \mathbf{Q}}(m+1) - V^{\lambda_j, \mathbf{Q}}(m)$,

$$\begin{aligned}
& \|\delta V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{B}[0, T]}^2 \\
& \leq 2K_1^2 K_2^2 \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_t^T \langle \delta \mathcal{Z}^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m-1)), d\mathcal{B}_s \rangle \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2 \\
& \quad \times \left\{ \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_t^T \langle \mathcal{Z}^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m)), d\mathcal{B}_s \rangle \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2 \right. \\
& \quad \left. + \sup_{\tau} \left\| E^{\mathbf{Q}} \left[\left| \int_t^T \langle \mathcal{Z}^{\lambda_j, \mathbf{Q}}(V^{\lambda_j, \mathbf{Q}}(m-1)), d\mathcal{B}_s \rangle \right| \middle| \mathcal{F}_{\tau} \right] \right\|_{\infty}^2 \right\} \\
& \leq 2K_1^2 K_2^2 \times 4 \|\delta V^{\lambda_j, \mathbf{Q}}(m-1)\|_{\mathfrak{B}[0, T]}^2 \\
& \quad \times 2 \left(8 \|\xi_j\|^2 + 4 \|V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{B}[0, T]}^2 + 4 \|V^{\lambda_j, \mathbf{Q}}(m-1)\|_{\mathfrak{B}[0, T]}^2 \right) \\
& \leq \frac{1}{4} \|\delta V^{\lambda_j, \mathbf{Q}}(m-1)\|_{\mathfrak{B}[0, T]}^2.
\end{aligned}$$

We iterate the above inequality, and obtain

$$\|\delta V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{B}[0, T]} \leq \left(\frac{1}{2} \right)^m \|V^{\lambda_j, \mathbf{Q}}(1)\|_{\mathfrak{B}[0, T]} \leq \left(\frac{1}{2} \right)^m \frac{1}{32K_1 K_2}.$$

Hence, for any natural number p ,

$$\begin{aligned}
& \|V^{\lambda_j, \mathbf{Q}}(m+p) - V^{\lambda_j, \mathbf{Q}}(m)\|_{\mathfrak{B}[0, T]} \\
& \leq \sum_{j=1}^p \|V^{\lambda_j, \mathbf{Q}}(m+j) - V^{\lambda_j, \mathbf{Q}}(m+j-1)\|_{\mathfrak{B}[0, T]} \\
& \leq \left(\frac{1}{2} \right)^{m-1} \times \frac{1}{32K_1 K_2}.
\end{aligned}$$

Letting $m \uparrow \infty$, we deduce that $\{V^{\lambda_j, \mathbf{Q}}(m)\}_{m \geq 0}$ is a Cauchy sequence, and converges to some limit $V^{\lambda_j, \mathbf{Q}}$. On the other hand, letting $p \uparrow \infty$, we obtain the convergence rate. ■

3.2 Nonlinear Girsanov's Transformation

The crucial step for our pseudo linear pricing rule (3.3) is to solve functional differential equation (3.2) in order to obtain $V^{\lambda, \mathbf{P}}$. Our second linear approximation for the utility indifference price \mathfrak{C}_t^{λ} is based on a nonlinear version of Girsanov's transformation in order to vanish the driver $F_t(z)$ of (3.2). Such an idea has been known in the BSDE literature. For example, it appears as Proposition 11 in Mania and Schweizer [30] and measure solutions of BSDEs in Ankirchner et al [1], where they model the terminal data (the payoff) as a general random variable. In contrast, as we specify the dynamics of the underlying and the payoff structure, a coupled FBSDE appears naturally.

The intuitive idea is as follows: Note that in Corollary 3.2, if we choose N :

$$\begin{aligned} N &= \int_0^\cdot \frac{\gamma}{2} \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}), d\mathcal{B}_s \rangle \\ &\quad - \int_0^\cdot \frac{\gamma}{2\|\sigma_s^P\|^2} \left\{ \langle \sigma_s^P, \mathcal{Z}_s^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}}) \rangle - \frac{2\mu_s^P}{\gamma} \right\} \langle \sigma_s^P, d\mathcal{B}_s \rangle, \end{aligned} \quad (3.9)$$

then the driver of functional differential equation (3.4) will vanish, and $V_t^{\lambda, \mathbf{Q}} = 0$ for $t \in [0, T]$. It seems that $V^{\lambda, \mathbf{P}}$ can be easily obtained by (3.6) and (3.4):

$$V_t^{\lambda, \mathbf{P}} = 0 - \int_0^t \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0), d[\mathcal{B}, N]_s \rangle = 0 + \int_0^t F_s(\mathcal{Z}_s^{\lambda, \mathbf{Q}}(0)) ds.$$

However, in this situation, the stochastic processes $\mathcal{Z}^{\lambda, \mathbf{Q}}(0)$ and \mathcal{S} depend on each other as a loop:

$$\int_t^T \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0), d\mathcal{B}_s \rangle = \left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) - E^{\mathbf{Q}} \left[\left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) \middle| \mathcal{F}_t \right], \quad (3.10)$$

and $\mathcal{S} = (S^1, \dots, S^d)$ given by

$$S_t^i = S_0^i + \int_0^t S_s^i (\mu_s^i ds - \langle \sigma_s^i, d[\mathcal{B}, N]_s \rangle + \langle \sigma_s^i, d\mathcal{B}_s \rangle). \quad (3.11)$$

Hence, we must solve (3.11) as a functional differential equation, which depends on $\mathcal{Z}^{\lambda, \mathbf{Q}}(0)$ as a functional of the whole path of \mathcal{S} . Note that (3.10) and (3.11) also form a special case of coupled FBSDEs, if we introduce a backward process Y^λ as conditional expectation:

$$Y_t^\lambda = E^{\mathbf{Q}} \left[\left(\lambda g(\mathcal{S}_\cdot) + \int_0^T \theta_s ds \right) \middle| \mathcal{F}_t \right].$$

With the help of such a nonlinear Girsanov's transformation, the remaining task is to solve functional differential equation (3.11). In the following, we work out its solution in a special Markovian setting:

- **Assumption (A4):** All the coefficients are deterministic, and the payoff $g(\cdot)$ is a positive bounded function satisfying:

$$|g(\mathcal{S}_T) - g(\bar{\mathcal{S}}_T)| \leq \frac{K_3}{\lambda} \sum_{i=1}^d |\ln S_T^i - \ln \bar{S}_T^i|.$$

A typical example that we keep in mind is $g(s) = \min(K, s)$ for some constant $K > 0$. Under Assumptions (A1)-(A4), the conditional expectation Y_t^λ can be written as a function of time t and the price process \mathcal{S}_t : $Y_t^\lambda = Y^\lambda(t, \mathcal{S}_t)$. If we define the log process $\mathcal{X}_t^i = \ln S_t^i$ for $i = 1, \dots, d$, then $Y^\lambda(t, e^{\mathcal{X}_t})$ is uniformly Lipschitz continuous in $\mathcal{X}_t = (\mathcal{X}_t^1, \dots, \mathcal{X}_t^d)$ (see Theorem 2.9 of Delarue [14]). We denote such a Lipschitz constant still as K_3 .

We first make a partition of $[0, T]$ as follows: $0 = t_0 < t_1 < \dots < t_J = T$ such that

$$\max_{1 \leq j \leq J} \Delta_j = \max_{1 \leq j \leq J} |t_j - t_{j-1}| \leq \frac{1}{8K_3^2 K_4},$$

where the definition of K_4 is given in the proof of the following Proposition 3.4. On each interval $[t_{j-1}, t_j]$, functional differential equation (3.11) can be reformulated in terms of \mathcal{X} :

$$\mathcal{X}_t^i = \mathcal{X}_{t_{j-1}}^i + \int_{t_{j-1}}^t (\mu_s^i - \frac{1}{2} \|\sigma_s^i\|^2) ds - \langle \sigma_s^i, d[\mathcal{B}, N]_s \rangle + \langle \sigma_s^i, d\mathcal{B}_s \rangle. \quad (3.12)$$

For functional differential equation (3.12), we can give the following linear approximation for its solution \mathcal{X} .

Define the Banach space $\mathfrak{S}([t_{j-1}, t_j]; \mathbb{R}^d)$ for the continuous and \mathcal{F}_t -adapted processes valued in \mathbb{R}^d , endowed with the norm:

$$\|\mathcal{X}\|_{\mathfrak{S}[t_{j-1}, t_j]} = E \left\{ \sup_{t \in [t_{j-1}, t_j]} \|\mathcal{X}_t\|^2 \right\}^{1/2}.$$

Proposition 3.4 *Let $\mathcal{B} = (B^1, \dots, B^d)$ be a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$. For fixed j where $1 \leq j \leq J$, suppose that we have solved functional differential equation (3.12) and obtained its solution \mathcal{X}_t for $t \in [t_j, t_{j+1}], \dots, [t_{J-1}, T]$, so that we have a Lipschitz continuous function $Y^\lambda(t_j, e^x)$ at time t_j .*

Define the sequence $\{\mathcal{X}(m)\}_{m \geq 0}$ on $[t_{j-1}, t_j]$ iteratively: $\mathcal{X}(0) = x$,

$$\mathcal{X}_t^i(m+1) = x^i + \int_{t_{j-1}}^t (\mu_s^i - \frac{1}{2} \|\sigma_s^i\|^2) ds - \langle \sigma_s^i, d[\mathcal{B}, N(m)]_s \rangle + \langle \sigma_s^i, d\mathcal{B}_s \rangle,$$

where $N(m)$ is given by (3.9) with $\mathcal{Z}^{\lambda, \mathbf{Q}}(V^{\lambda, \mathbf{Q}})$ replaced by $\mathcal{Z}^{\lambda, \mathbf{Q}}(0, m)$:

$$\int_t^{t_j} \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0, m), d\mathcal{B}_s \rangle = Y^\lambda(t_j, e^{\mathcal{X}_{t_j}(m)}) - E^{\mathbf{Q}} \left[Y^\lambda(t_j, e^{\mathcal{X}_{t_j}(m)}) \mid \mathcal{F}_t \right].$$

Then $\{\mathcal{X}(m)\}_{m \geq 0}$ converges to some \mathcal{X} in the space $\mathfrak{S}([t_{j-1}, t_j]; \mathbb{R}^d)$ with the convergence rate

$$\|\mathcal{X} - \mathcal{X}(m)\|_{\mathfrak{S}[0, T]} \leq \left(\frac{1}{2} \right)^{m-1} \times \|\mathcal{X}(1) - \mathcal{X}(0)\|_{\mathfrak{S}[t_{j-1}, t_j]},$$

from which we get a Lipschitz continuous function at time t_{j-1}

$$Y^\lambda(t_{j-1}, e^{\mathcal{X}_{t_{j-1}}}) = E^{\mathbf{Q}} \left[Y^\lambda(t_j, e^{\mathcal{X}_{t_j}}) \mid \mathcal{F}_{t_{j-1}} \right],$$

and $\mathcal{Z}^{\lambda, \mathbf{Q}}(0)$ on $[t_{j-1}, t_j]$

$$\left\langle \int_{t_{j-1}}^{t_j} \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0), d\mathcal{B}_s \right\rangle = Y^\lambda(t_j, e^{\mathcal{X}_{t_j}}) - Y^\lambda(t_{j-1}, e^{\mathcal{X}_{t_{j-1}}}).$$

Finally, N defined by (3.9) is a \mathbf{Q} -BMO martingale, and $V^{\lambda, \mathbf{P}}$ is obtained by (3.6) and (3.4):

$$V_t^{\lambda, \mathbf{P}} = \int_0^t F_s(\mathcal{Z}_s^{\lambda, \mathbf{Q}}(0)) ds.$$

Proof. The proof is similar to the proof of Proposition 3.3, so we only sketch it. For fixed j , it is easy to verify that $\mathcal{X}(m+1) \in \mathcal{S}[t_{j-1}, t_j]$, if $\mathcal{X}(m) \in \mathcal{S}[t_{j-1}, t_j]$.

Next, consider the difference $\delta\mathcal{X}(m) = \mathcal{X}(m+1) - \mathcal{X}(m)$,

$$\begin{aligned}
& \|\delta\mathcal{X}(m)\|_{\mathfrak{S}[t_{j-1}, t_j]}^2 \\
&= E^{\mathbf{Q}} \left\{ \sup_{t \in [t_{j-1}, t_j]} \sum_{i=1}^d \left| \int_{t_{j-1}}^t \langle \sigma_s^i, d[\mathcal{B}, \delta N(m-1)]_s \rangle \right|^2 \right\} \\
&= E^{\mathbf{Q}} \left\{ \sup_t \sum_i \left| \int_{t_{j-1}}^t \left(\frac{\gamma}{2} \langle \sigma_s^i, \delta \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0, m-1) \rangle \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{\gamma \langle \sigma_s^i, \sigma_s^P \rangle}{2 \|\sigma_s^P\|} \langle \sigma_s^P, \delta \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0, m-1) \rangle \right) ds \right|^2 \right\} \\
&\leq K_4 \Delta_j E^{\mathbf{Q}} \left\{ \int_{t_{j-1}}^{t_j} \|\delta \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0, m-1)\|^2 ds \right\} \\
&= K_4 \Delta_j E^{\mathbf{Q}} \left\{ \left| \int_{t_{j-1}}^{t_j} \langle \delta \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0, m-1), d\mathcal{B}_s \rangle \right|^2 \right\} \\
&\leq 2K_4 \Delta_j E^{\mathbf{Q}} \left\{ \left| Y^\lambda(t_j, e^{\mathcal{X}_{t_j}(m)}) - Y^\lambda(t_j, e^{\mathcal{X}_{t_j}(m-1)}) \right|^2 \right\} \\
&\leq 2K_3^2 K_4 \Delta_j \|\delta\mathcal{X}(m-1)\|_{\mathfrak{S}[t_{j-1}, t_j]}^2 \leq \frac{1}{4} \|\delta\mathcal{X}(m-1)\|_{\mathfrak{S}[t_{j-1}, t_j]}^2.
\end{aligned}$$

We iterate the above inequality and obtain

$$\|\delta\mathcal{X}(m)\|_{\mathfrak{S}[t_{j-1}, t_j]} \leq \left(\frac{1}{2}\right)^m \|\mathcal{X}(1) - \mathcal{X}(0)\|_{\mathfrak{S}[t_{j-1}, t_j]}.$$

Hence, for any natural number p ,

$$\begin{aligned}
\|\mathcal{X}(m+p) - \mathcal{X}(m)\|_{\mathfrak{S}[t_{j-1}, t_j]} &\leq \sum_{j=1}^p \|\mathcal{X}(m+j) - \mathcal{X}(m+j-1)\|_{\mathfrak{S}[t_{j-1}, t_j]} \\
&\leq \left(\frac{1}{2}\right)^{m-1} \times \|\mathcal{X}(1) - \mathcal{X}(0)\|_{\mathfrak{S}[t_{j-1}, t_j]}.
\end{aligned}$$

Letting $m \uparrow \infty$, we deduce that $\{\mathcal{X}(m)\}_{m \geq 0}$ is a Cauchy sequence, and converges to some limit \mathcal{X} . On the other hand, letting $p \uparrow \infty$, we obtain the convergence rate.

The rest of the proof is to verify that N defined by (3.9) is a \mathbf{Q} -BMO martingale, which follows from the \mathbf{Q} -BMO martingale property of

$$\int_0^\cdot \langle \mathcal{Z}_s^{\lambda, \mathbf{Q}}(0), d\mathcal{B}_s \rangle.$$

■

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